光電子學導論

(3) Fundamentals of Laser Theory

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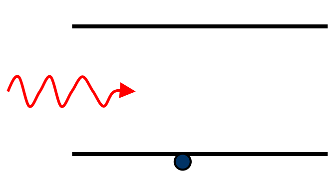
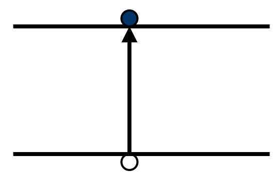
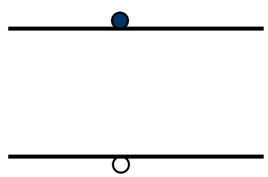
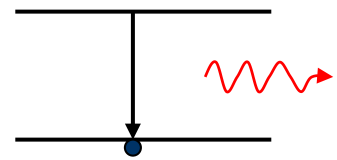
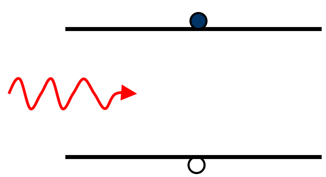
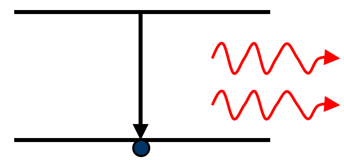
What Is LASER?

- LASER is the abbreviation of 『Light Amplification by Stimulated Emission of Radiation』
- Taiwan ⇨ 雷射 (取其音)
- China ⇨ 激光 (取其意)

MASER and LASER

- In 1951, Charles Hard Townes conceived a new way to create intense, precise beams of coherent radiation, for which he invented the acronym MASER (for Microwave Amplification by Stimulated Emission of Radiation). When the same principle was applied to higher frequencies, the term LASER was used (the word "light" substituting for the word "microwave").
- During 1953, Charles Hard Townes , James P. Gordon, and Herbert J. Zeiger built the first ammonia maser at Columbia University.
- In 1958, Arthur Leonard Schawlow and Charles Hard Townes published a paper on the laser effect in optical spectral range (Infrared and Optical Masers, Phys. Rev. 112, pp. 1940–1949, December 15, 1958).
- On May 16, 1960, Ruby laser was successfully demonstrated by Theodore Harold Maiman, who was with the Hughes Aircraft Company (Malibu, California), which was the first laser ever demonstrated. In a July 7, 1960 press conference in Manhattan, Maiman announced the laser to the world. Paper was then published in Nature (Stimulated Optical Radiation in Ruby, Nature 187, pp. 493–494, August 6, 1960).
- Charles Hard Townes shared the 1964 Nobel Prize in Physics with Nikolay Basov and Alexander Prokhorov. Arthur Leonard Schawlow shared the 1981 Nobel Prize in Physics with Nicolaas Bloembergen and Kai Siegbahn.

Interaction Between Photon and Electron

Mechanism	Before		After	Device
Absorption		High Low		Solar Cell Detector
Spontaneous Emission		High Low		LED OLED
Stimulated Emission		High Low		Laser

$$\boxed{\lambda\ (\,\mathrm{nm}\,) \approx 1240 \,/\, E_g\ (\,\mathrm{eV}\,)}$$

Fundamentals of Laser Light

- Laser light is electromagnetic wave in nature. It travels with a speed of 3×10^8 m/sec in free space (vacuum).
- Laser light has very narrow spectral width, which is close to the characteristic of monochromatic light.
- Laser light has very high degree of coherence so that all the photons travel in the same direction with the same phase.

Characteristics of Laser Light

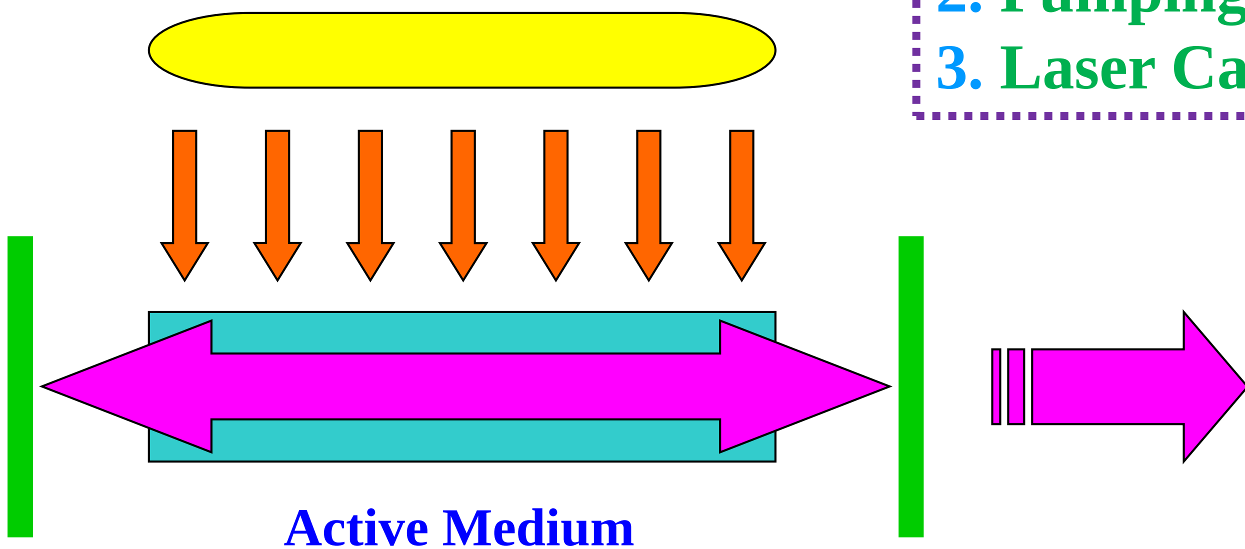
- Owing to the coherence characteristic of laser light, very high laser power can be focused in very small area, which will result in extremely high intensity.
- Even the relatively low power emitted by a laser pointer could cause severe damage to the human eyes.
- The green light emitted from a frequency-doubled Nd:YAG laser pointer is quite dangerous, especially to the human eyes.

Applications of Lasers

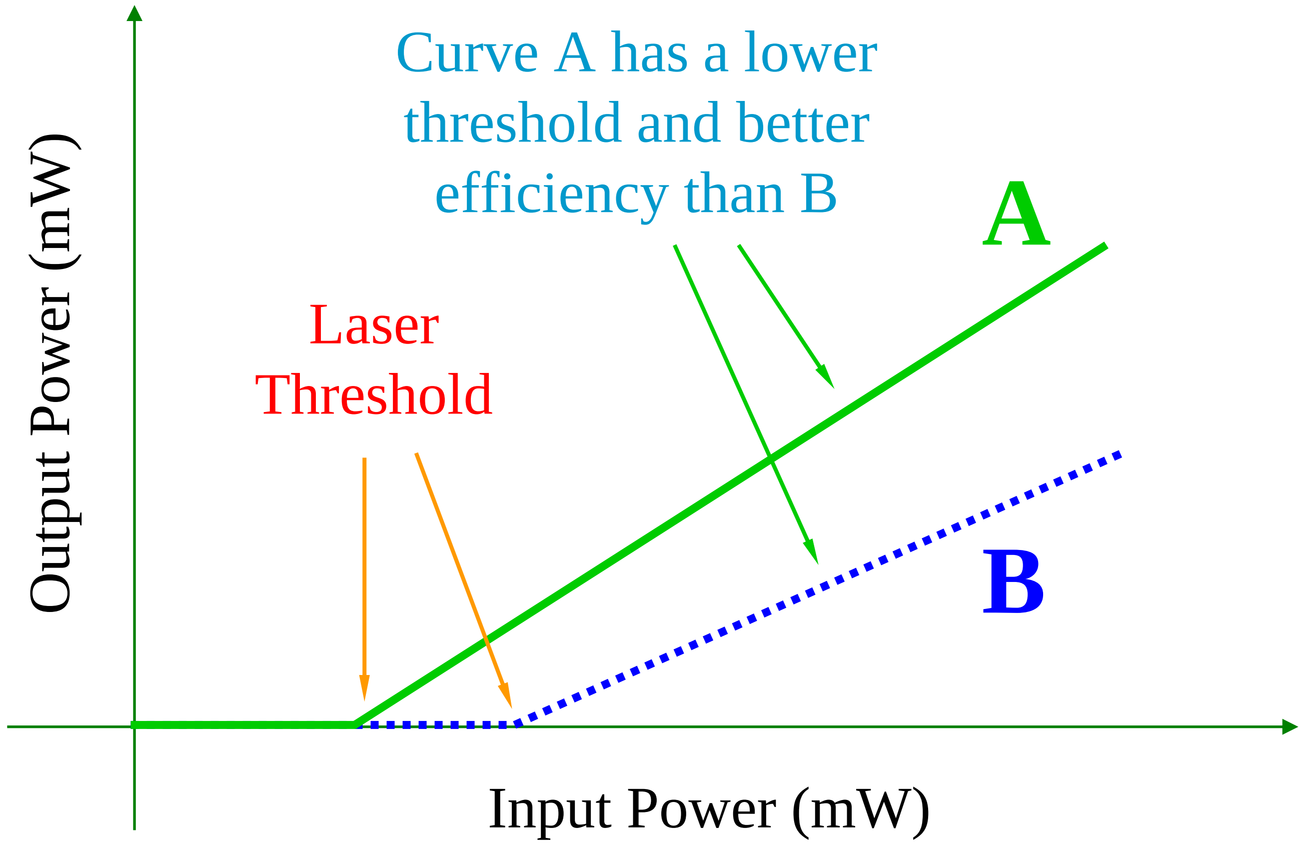
- **Science:** Interference, Diffraction, Holograms, Photoluminescence, Biology, etc.
- **Medicine:** Ophthalmology, Surgery, Dermatology, etc.
- **Industry:** Cutting, Drilling, Welding, etc.
- **Communication:** Optical Fiber Communication, Internet, etc.
- **Daily Lives:** CD/DVD, Laser Pointer, Bar-Code Reader, etc.
- **Military:** Laser Ranging, Laser Designator, Laser Gyro, High-Power Laser Weapon, Purification of Uranium, etc.

Fundamental Elements of a Laser System

Laser System:
1. **Active Medium**
2. **Pumping System**
3. **Laser Cavity**



Typical Performance Curve (Output-Input Relation)



Fundamental Characteristics of Laser Performance

- Threshold condition for laser action: $R_1R_2e^{2gl}e^{-2\alpha l} = 1$, or $R_1R_2e^{(g-\alpha)2l} = 1$, where R_1, R_2 = mirror reflectivities, g = gain coefficient, α = absorption coefficient.
  If $R_1 \cong 100\%$, $R_2 = R$, $L = 2\alpha l$, then $2gl = L - \ln(R) \cong L + T$ when R is close to unity.
  (note: T, equal to 1–R, is the transitivity of the output-coupling mirror)
- Gain saturation: gain coefficient, g, is a function of laser intensity

  $$g = \frac{g_0}{1 + I / I_s}, \text{ where } I_s = \frac{h\nu}{\sigma_{21}\tau_f}$$

  g_0 = small-signal gain coefficient, I = intensity, I_s = saturation intensity,
  σ_{21} = emission cross-section, τ_f = fluorescence lifetime.
- The laser performance curve shown in last figure can be characterized by:
  $P_{out} = \sigma_s (P_{in} - P_{th})$, where $\sigma_s = \eta_p\eta_t\eta_a\eta_q\eta_s\eta_b\eta_c$.
  η_p: electrical input to pump source $\rightarrow$ useful pump radiation
  η_t: useful pump radiation $\rightarrow$ transferred to gain medium
  η_a: absorption of pump radiation by the gain medium
  η_q: number of photons contributing to laser emission / number of pump photons
  η_s: $h\nu_{laser} / h\nu_{pump}$ (= $\lambda_{pump} / \lambda_{laser}$, i.e., pump wavelength / laser wavelength)
  η_b: beam overlap efficiency (between pump and resonator mode)
  η_c: coupling efficiency [$\cong T / (T+L)$]

Excitation of an Atom

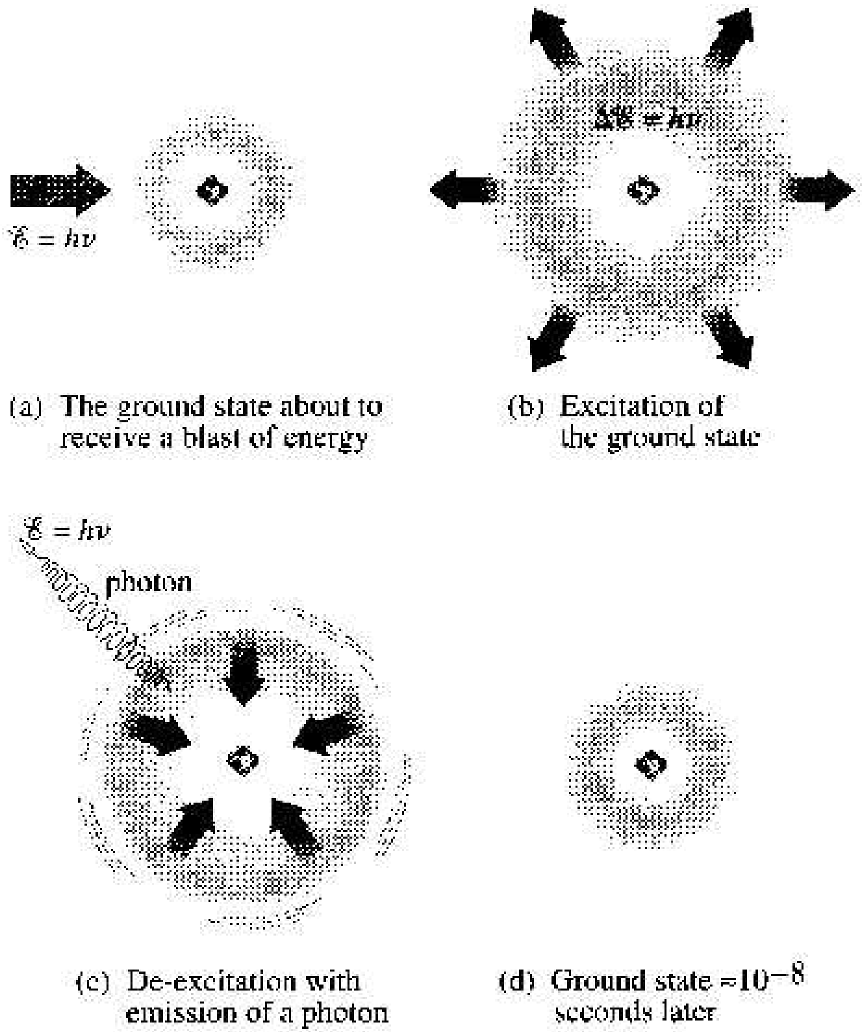


Figure 3.35 The excitation of an atom. (a) Energy in the amount $h\nu$ is delivered to the atom. (b) Since this matches the energy needed to reach an excited state, the atom absorbs the energy and attains a higher energy level. (c) With the emission of a photon, it drops back (d) and returns to the ground state in about 10^{-8} s.

Black Body Radiation

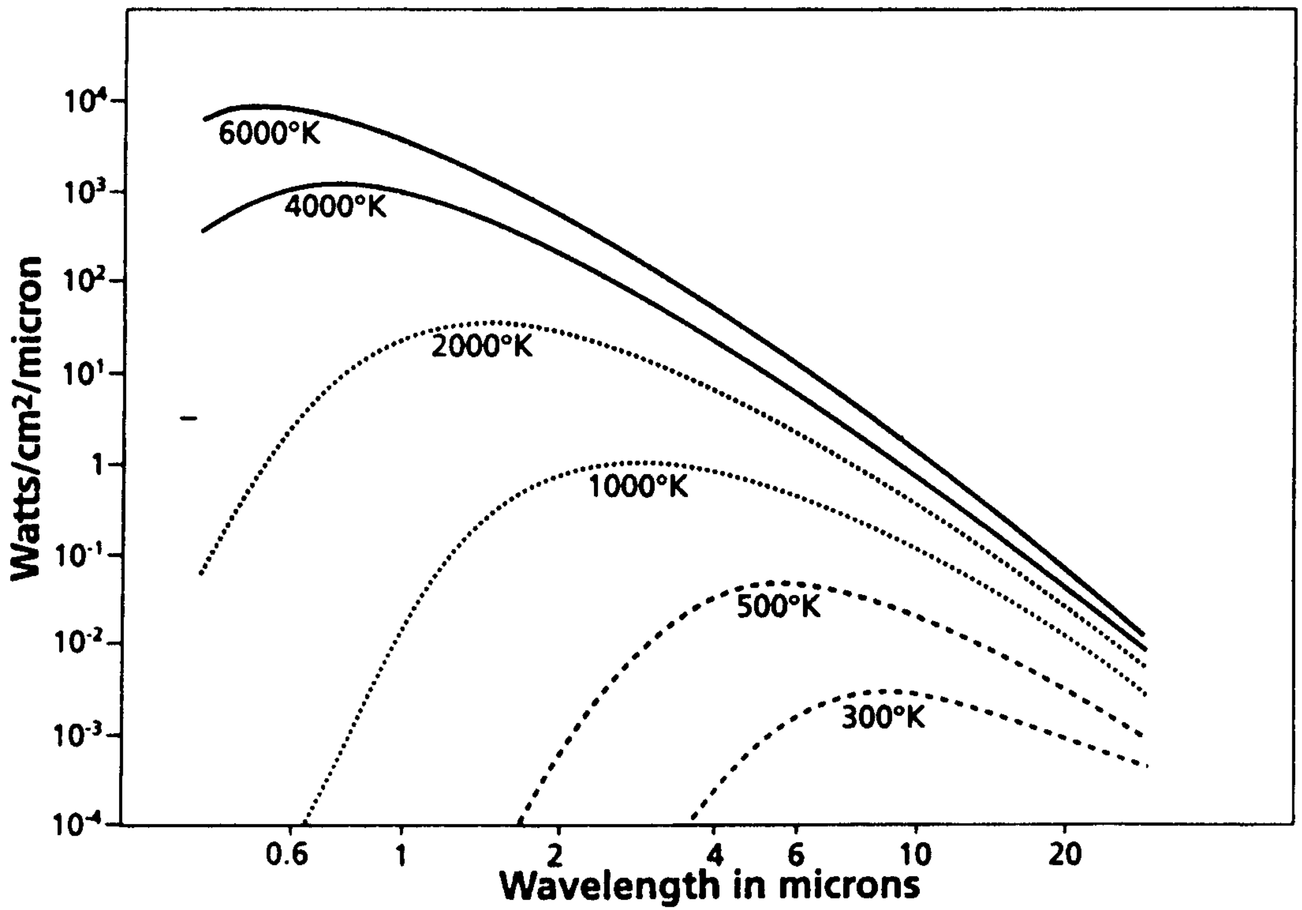


Figure 8.3 Blackbody spectral radiant emittance for various temperatures. Reprinted with permission of Cal Sensors.

Energy Levels and Electron Population Distribution

1) Many solids radiate like a blackbody and have following characteristics:

a) total radiation power density, $w = \sigma T^4$, where σ is a constant,

b) maximum radiation wavelength, $\lambda_{max}(\mu m) \cong 2893/T(°K)$.

2) Assuming that we have a large collection of similar atoms which are in thermal equilibrium at temperature *T*, Boltzmann statistics states that

$$N = N_0\, e^{-E/kT} \Rightarrow \frac{N_2}{N_1} = \frac{N_0\, e^{-E_2/kT}}{N_0\, e^{-E_1/kT}} = e^{-(E_2-E_1)/kT}$$

E_4 (N_4)

E_3 (N_3)

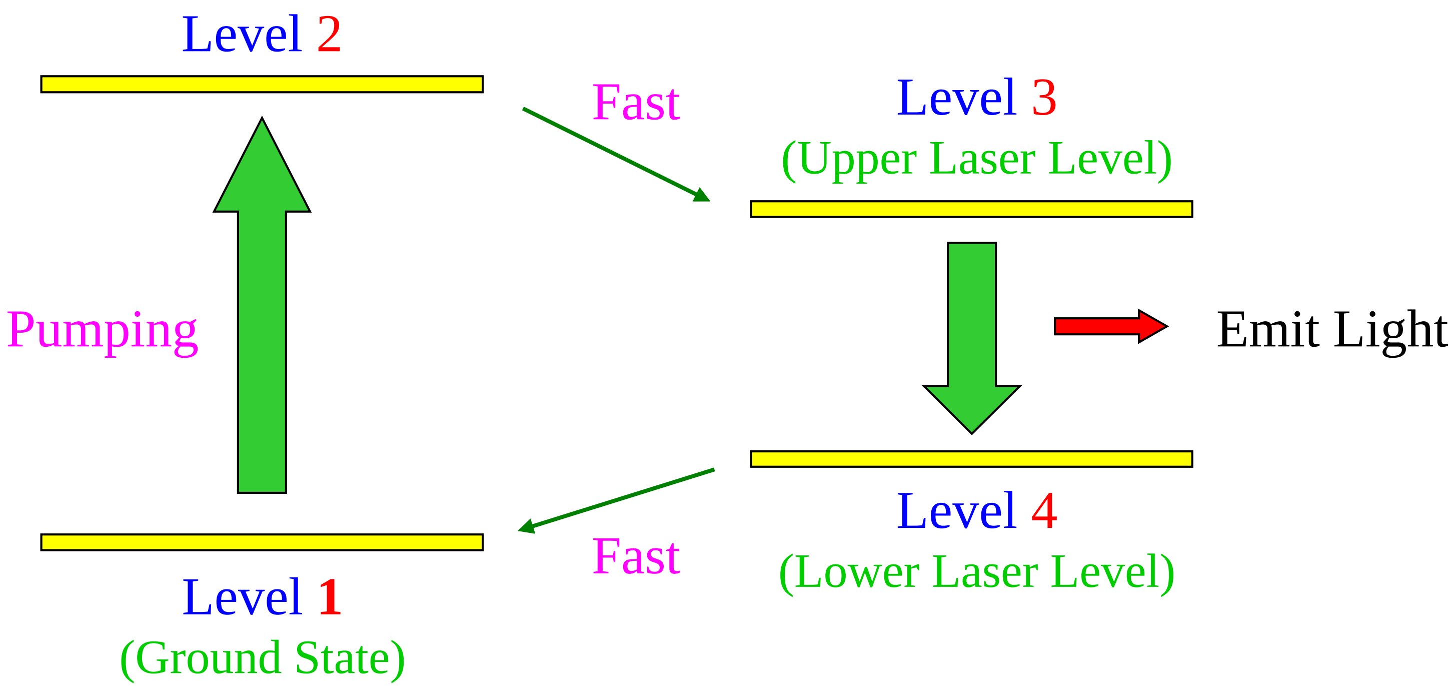
E_2 (N_2)

E_1 (N_1)

For example, for ruby laser: $\lambda = 694.3$ nm, $E_2 - E_1 = h\nu = hc/\lambda = 2.86\times10^{-19}$ J.
Since $k = 1.38\times10^{-23}$ J/K and $T = 300$ K, $N_2/N_1 \cong 10^{-32}$, which is a very small number.
It indicates that virtually all the atoms are in the ground state at room temperature.

3) Population inversion has to be obtained (for example, by using flashlamp, lasers, injection current, etc.) to activate laser action.

4-Level and 3-Level Laser Systems



- The dynamics of most lasers, including the semiconductor lasers, can be described by the four energy levels shown above. These lasers are called "4-level lasers" that usually have good laser efficiency.
- For some lasers (e.g., ruby) the lower laser level is in fact the ground state. These lasers are called "3-level lasers" that usually have relatively poor laser efficiency.

Rate Equations of a Laser System

$$\frac{dn}{dt} = K_g N_g n - \gamma_c n$$

$$\frac{dN_g}{dt} = R_p - \gamma_g N_g - \gamma K_g N_g n$$

γ : population reduction factor (= 1 for 4-level laser, 2 for 3-level laser)

n : photon number

N_g : laser population inversion

γ_c : cavity decay rate

γ_g : decay rate of upper laser level

R_p: pumping rate

K_g : coupling coefficient

Reference: A. E. Siegman, *Lasers* (1986)

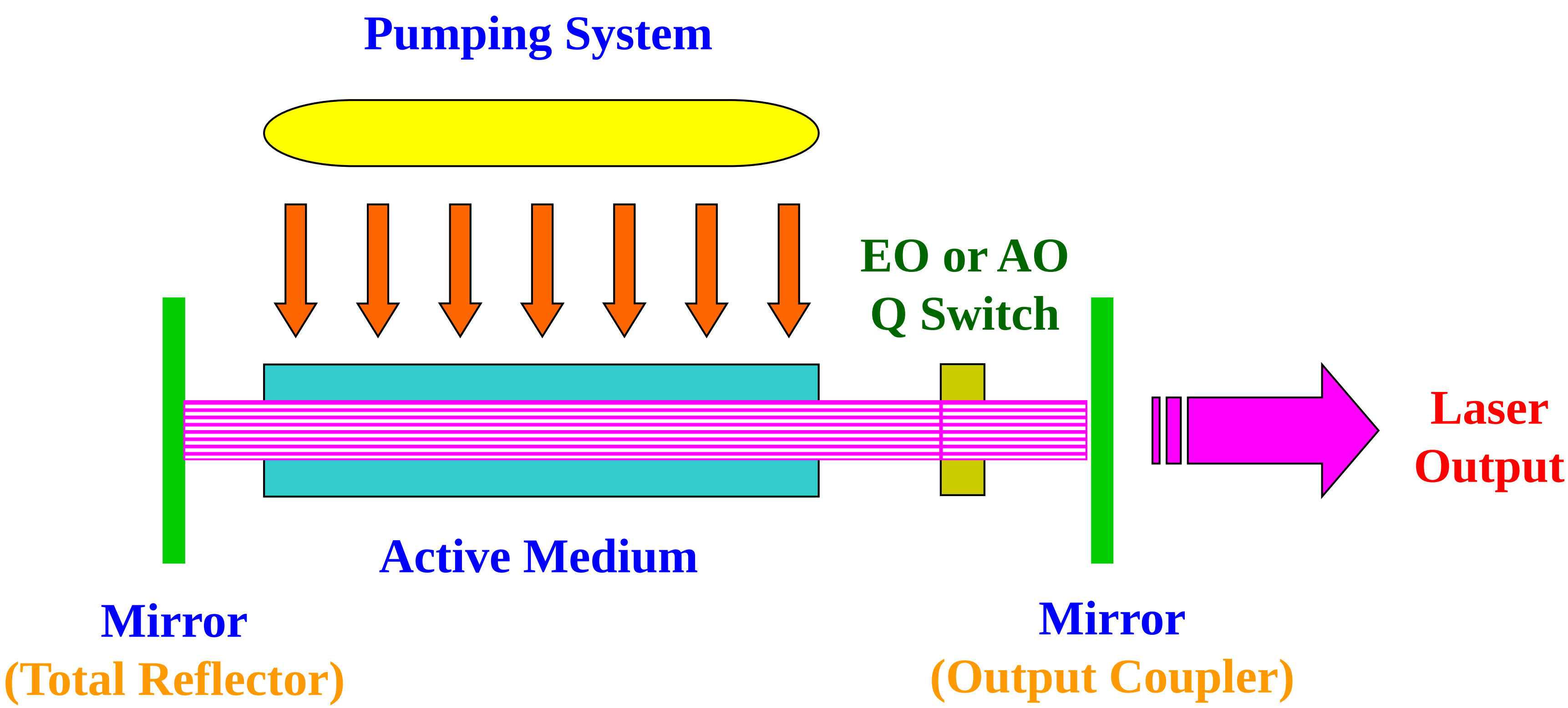
Q-Switching

- **In an optical resonator the quality factor Q is defined as the ratio of the energy stored in the laser cavity to the energy loss per cycle. Therefore, the quality factor of a laser resonator can be altered by varying the cavity loss.**
- **In the technique of Q-switching, energy is stored in the gain medium through optical pumping while the quality factor is lowered in a particular way, depending on the method used, to prevent from laser oscillation. After a high population inversion has been developed, the high cavity Q is then restored, or "switched", such that the stored energy can be released in very short time.**
- **The pulsewidth available in this manner is usually in the order of tens of nanoseconds and the peak power can be several orders higher than that of the ordinary long laser pulse.**
- **Q-switching of the solid-state lasers is important because it provides short duration optical pulses required for laser ranging, nonlinear studies, medicine and other important applications.**

Active Q-Switching and Passive Q-Switching

- In technique of active Q-switching, electro-optic Q-switches or acousto-optic Q-switches are used to obtain giant laser pulses. The generation of the Q-switched laser pulses can be controlled by the externally applied voltage.
- In technique of passive Q-switching, saturable absorber Q-switches such as the dye cells, semiconductor films, or solid-state crystals are utilized to obtain giant laser pulses.
- Dye cells and semiconductor materials suffer damage problems. On the other hand, solid-state crystals (such as Cr:YAG, Cr:forsterite, Cr:Y_2SiO_5 (Cr:YSO), Dy:CaF_2, etc.) are relatively robust (durable) and hence have better reliability than the traditional dye saturable absorbers.

Active Q-Switching



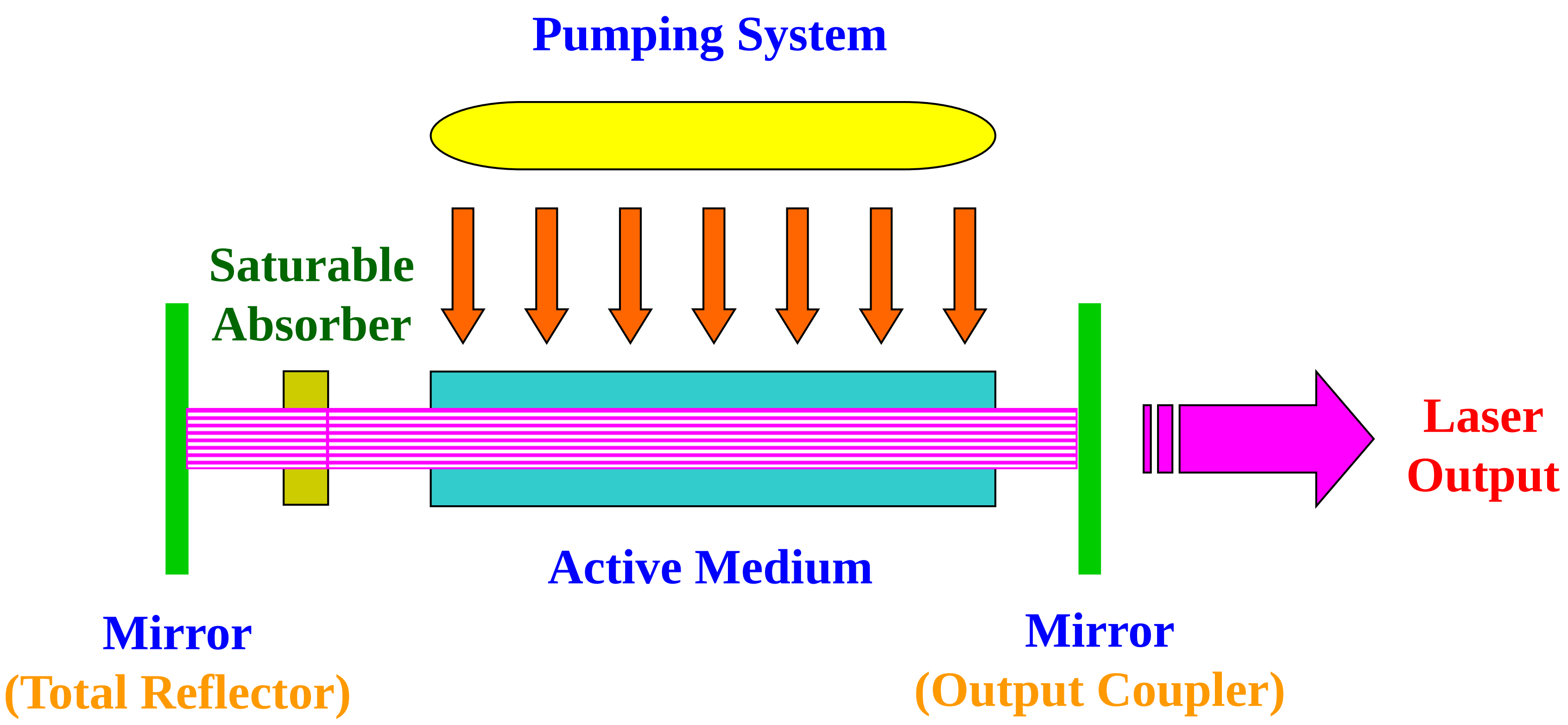
Electro-Optic (EO) Q-Switching

- Fast optical shutters can be made by using the electro-optic effect in some solid-state crystals, which become birefringent under the influence of an externally applied electric field, with polarizing element inside the laser cavity.
- The low-Q state is achieved by applying a voltage to the Q-switch such that the polarization of a linearly polarized laser light is rotated by 90° after a round trip inside the laser cavity. The laser is prohibited from oscillation in this state due to the lack of optical feedback. The high-Q state is achieved by turning off the applied voltage such that a linearly polarized laser light can get through the optical elements without loss.
- The pulsewidth of the Q-switched laser output is usually between 10 and 25 ns with electro-optic Q-switches. Typical output energy is 100-250 mJ for an Nd:YAG laser and about 100 mJ for a ruby laser with a 3-inch-long laser rod.
- Electro-optic Q-switching is effective. However, besides the expensive crystal and optical elements this method requires a very fast rising high-voltage pulse source which adds cost and complexity to the overall Q-switched laser system. In addition, this approach needs several optical elements inside the cavity which introduce extra losses and may subject to optical damage.

Acousto-Optic (AO) Q-Switching

- Another approach to active Q-switches is based on the acousto-optic effect. In acousto-optic Q-switches, an acoustic wave is typically launched into the Q-switch (usually fused silica) by a piezo-electric transducer. The Q-switch acts like a grating when the acoustic wave is present. When the light beam passes through the Q-switch a portion of the light is diffracted out of the beam by the grating, which thus results in a higher cavity loss or, equivalently, a lower quality factor. High cavity Q is achieved by turning off the acoustic wave.
- Compared with electro-optic Q-switching, acousto-optic Q-switching has low insertion loss and is convenient to use; however, it has relatively slow switching time and low hold-off ratio.
- A regular cw-pumped Nd:YAG laser can produce Q-switched laser pulses of about 40 KW peak power with an acousto-optic Q-switch. Typical Q-switched laser pulses are ~ 40 ns full width at half maximum (FWHM).

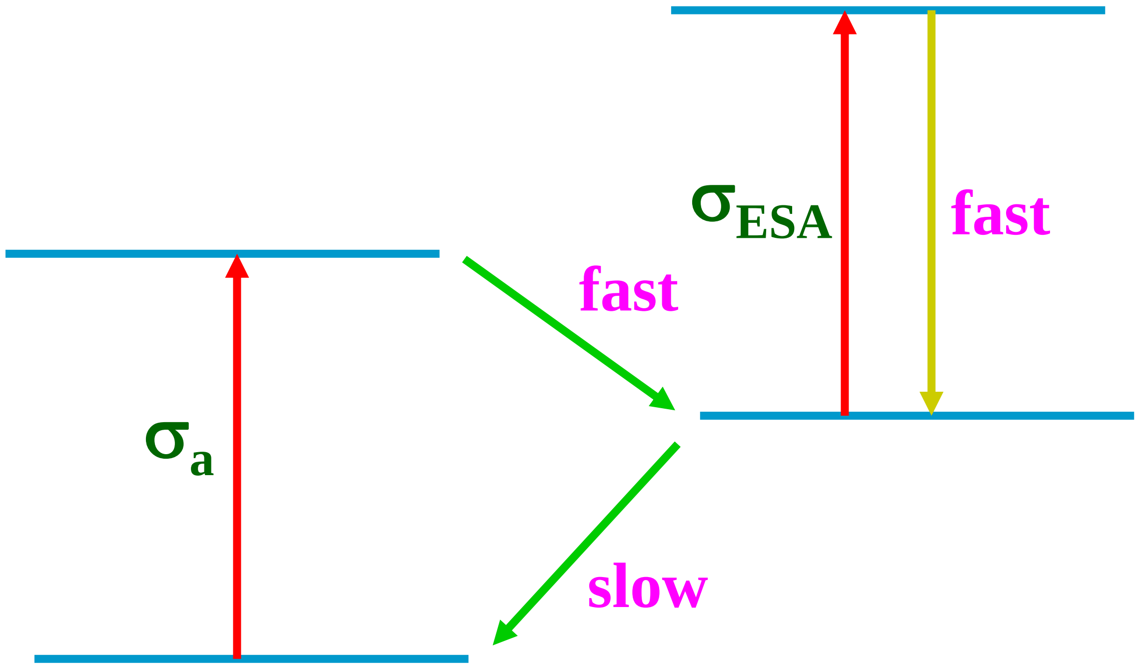
Passive Q-Switching



Comparison Between Active and Passive Q-Switching

- The active Q-switches, which employ either electro-optic or acousto-optic effect, have been widely used in many of the industrial applications because they operate reliably over an extended period of time and can be triggered at any moment within the pumping cycle. However, the overall laser system is rather complicated.
- Compared to the active Q-switching, passive Q-switching with saturable absorber is economical and simple because it requires less optical elements inside the laser cavity and no outside driving circuitry. Passive Q-switching is a better choice for those applications where compactness of the laser is a prime requirement.

Absorption at Higher Level of Saturable Absorber



- σ_a: Ground state absorption cross-section
- σ_{ESA}: Excited state absorption cross-section
- $\beta = \sigma_{ESA} / \sigma_a$

Laser Equations of Passive Q-Switching

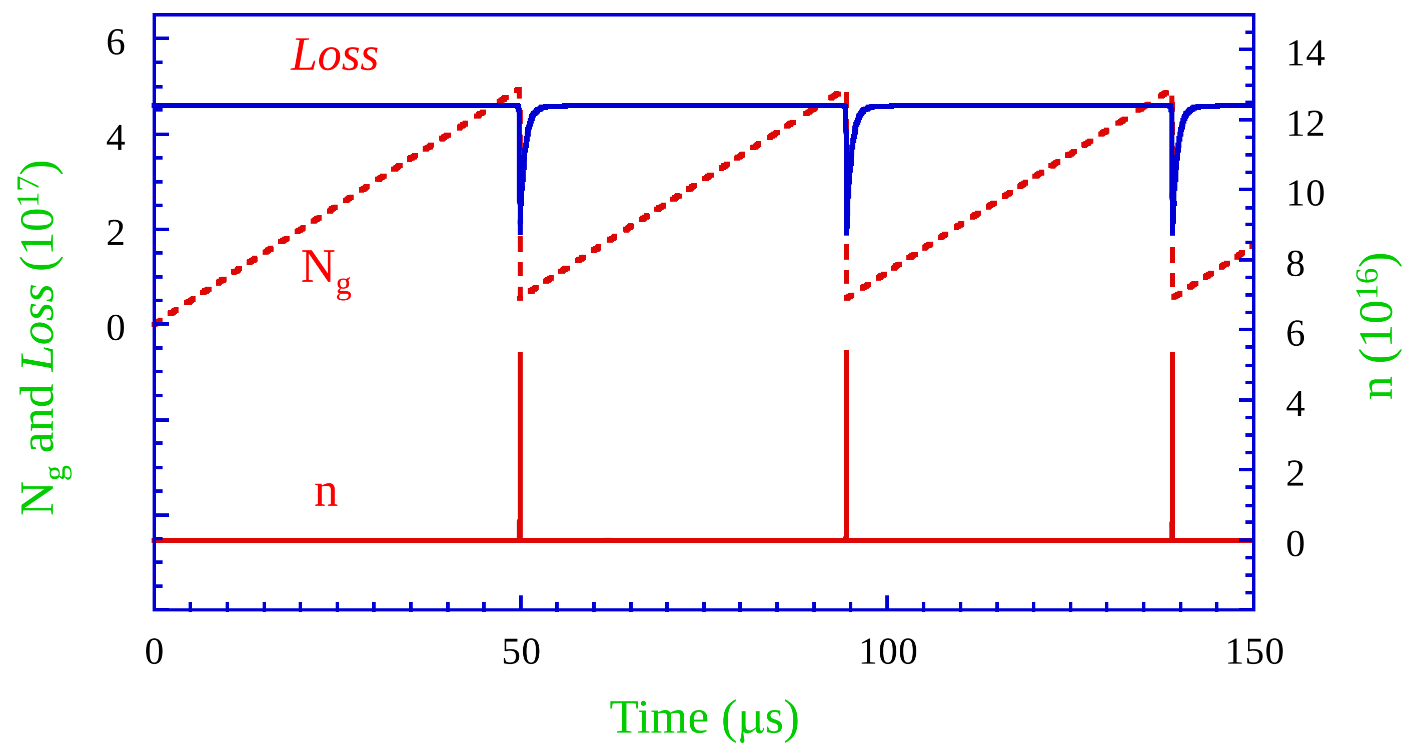
$$\frac{dn}{dt} = K_g N_g n - K_a N_a n - \beta K_a (N_{a0} - N_a) n - \gamma_c n$$

$$\frac{dN_g}{dt} = R_p - \gamma_g N_g - \gamma K_g N_g n$$

$$\frac{dN_a}{dt} = \gamma_a (N_{a0} - N_a) - K_a N_a n$$

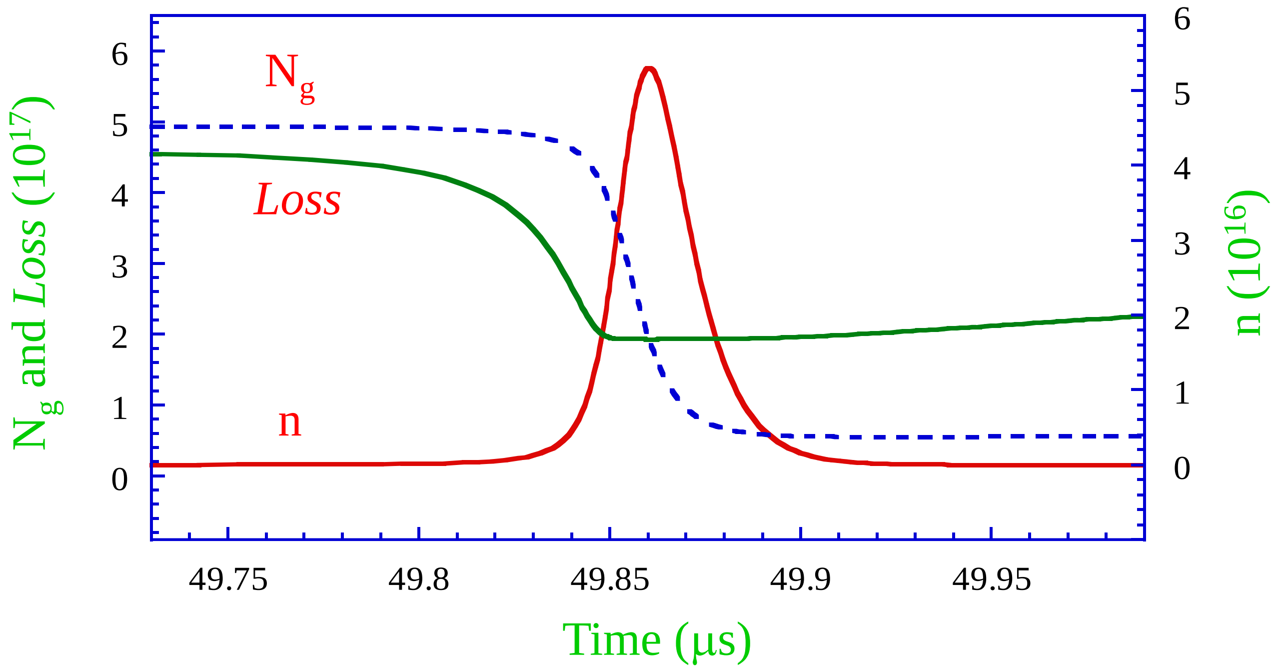
The population reduction factor, γ, equals to one for a four-level laser and two for a three-level laser. Other parameters are defined as following: n is the photon number in the laser cavity; N_g is the population inversion of the laser; N_a is the ground state population of the saturable absorber; N_{a0} is the initial value of N_a; γ_g is the effective decay rate of the upper laser level; γ_a is the effective relaxation rate of the saturable absorber; R_p is the pumping rate; γ_c is the cavity decay rate; K_g and K_a are coupling coefficients; and β is the ratio of the excited state absorption cross-section to the ground state absorption cross-section of the saturable absorber.

Ruby Laser with Cr:YSO Saturable Absorber



$$Loss \cong \frac{K_a N_a + \beta K_a (N_{a0} - N_a) + \gamma_c}{K_g}$$

Expanded Diagram Near Laser Pulse



- **When the photon number increases the loss decreases accordingly due to the bleaching effect of the Cr:YSO saturable absorber.**
- **The photon number reaches to the peak when N_g = *Loss*. Beyond this point the laser gain is smaller than the total loss and the Q-switched laser pulse dies out quickly while the laser population inversion decreases gradually to a minimum value.**

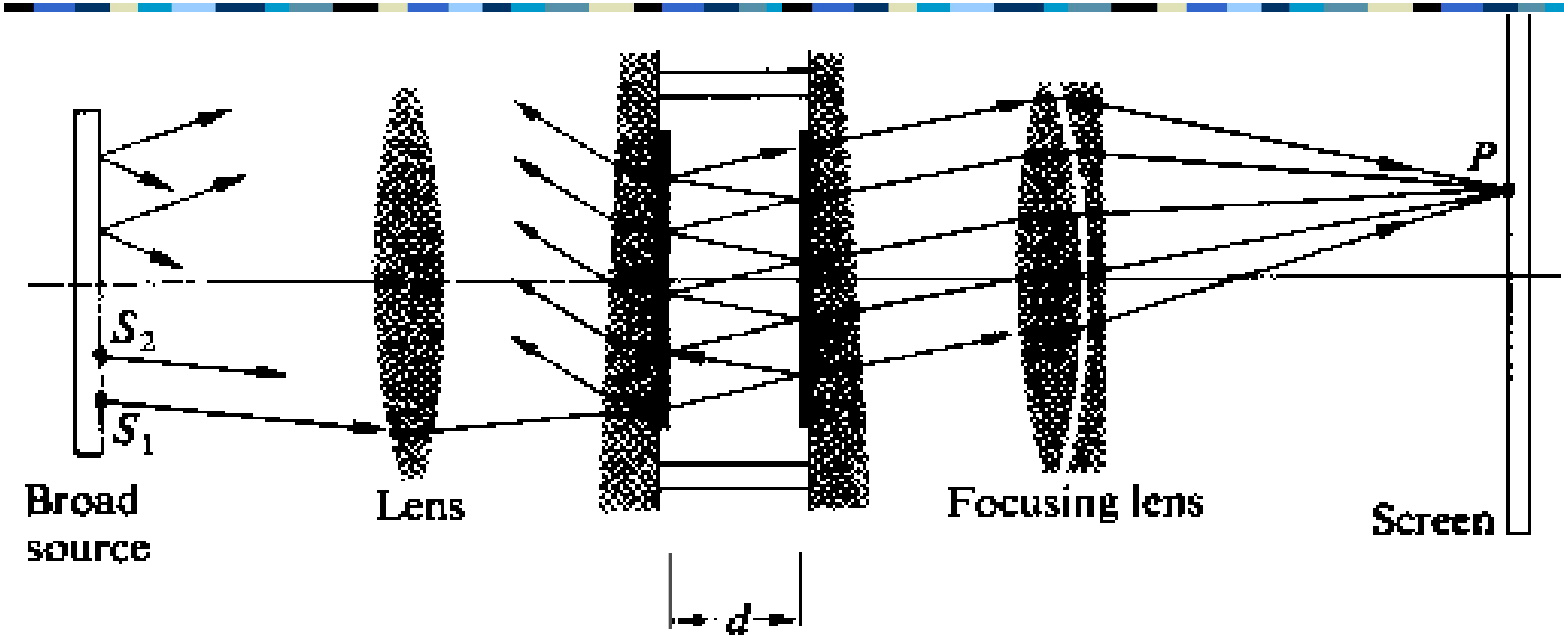
Mode-Locking

- **Ultrashort pulses** with pulse widths in the picosecond (10^{-12} sec) or femtosecond (10^{-15} sec) regime may be obtained from solid-state lasers by mode-locking.
- Employing this technique, which phase-locks the longitudinal modes of the laser, the pulse width is inversely related to the bandwidth of the laser emission. Therefore, **tunable solid-state lasers are perfect candidates for mode-locking**, although other solid-state lasers (e.g., Nd:YAG, etc.) may also be mode-locked to obtain short laser pulses.
- The output from laser oscillators is subject to strong fluctuations which originate from the interference of longitudinal laser modes with random phase relations. These random fluctuations can be transformed into a powerful well-defined single pulse circulating in the laser resonator by the introduction of a suitable nonlinearity (i.e., **passive mode-locking**) or by an externally driven optical modulator (i.e., **active mode-locking**).

Characteristics of Mode-Locked Lasers

- **Longitudinal modes: $L = (\lambda/2) \times n$, $n = 1, 2, 3, \ldots$**
  $\Rightarrow L = (C/2\nu) \times n \Rightarrow \nu = (C/2L) \times n \Rightarrow \Delta\nu = C/2L$ (Hz) = longitudinal mode spacing in frequency (i.e., Free Spectral Range, FSR)
- **$1/\Delta\nu = 2L/C = \Delta T$ (sec) = round-trip transit time**
- **For a mode-locked laser, pulse width = $\tau_p \cong 1/\Delta\nu_L$, $\Delta\nu_L$ = gain bandwidth**
- **Assuming that we have N longitudinal modes, $\Delta\nu_L = N \times \Delta\nu = N/\Delta T \Rightarrow \tau_p \cong 1/\Delta\nu_L = \Delta T/N$**
- **Peak Power = $P_{peak} = N \times P_{avg}$ (W), P_{avg} = Average Power**
- **Example: for a typical Nd:YAG laser, $\lambda = 1.064$ μm, $L = 1.2$ m, $\Delta T = 8$ ns, $\Delta\nu = 125$ MHz, $N = 400$ (modes), $\tau_p \cong 20$ ps.**

Fabry-Perot (Charles Fabry and Alfred Perot) Interferometer (Etalon)



Fabry-Perot interferometer **consists of two plane, parallel, highly reflecting surfaces separated by some distance *d*. When the mirrors are held fixed and adjusted for parallelism by screwing down on some sort of spacer, it's said to be an *etalon*.**

If the two surfaces of a single *quartz plate* are appropriately polished and silvered, it too will serve as an etalon.

Multiple-Beam Interference
(Etalon, or Fabry-Perot Resonator)

$$E_T = E_0 t^2 + E_0 t^2 r^2 e^{i\delta} + E_0 t^2 r^4 e^{2i\delta} + \dots$$

$$\text{where} \quad \delta = 2kd\cos\theta = \frac{4\pi}{\lambda} d\cos\theta$$

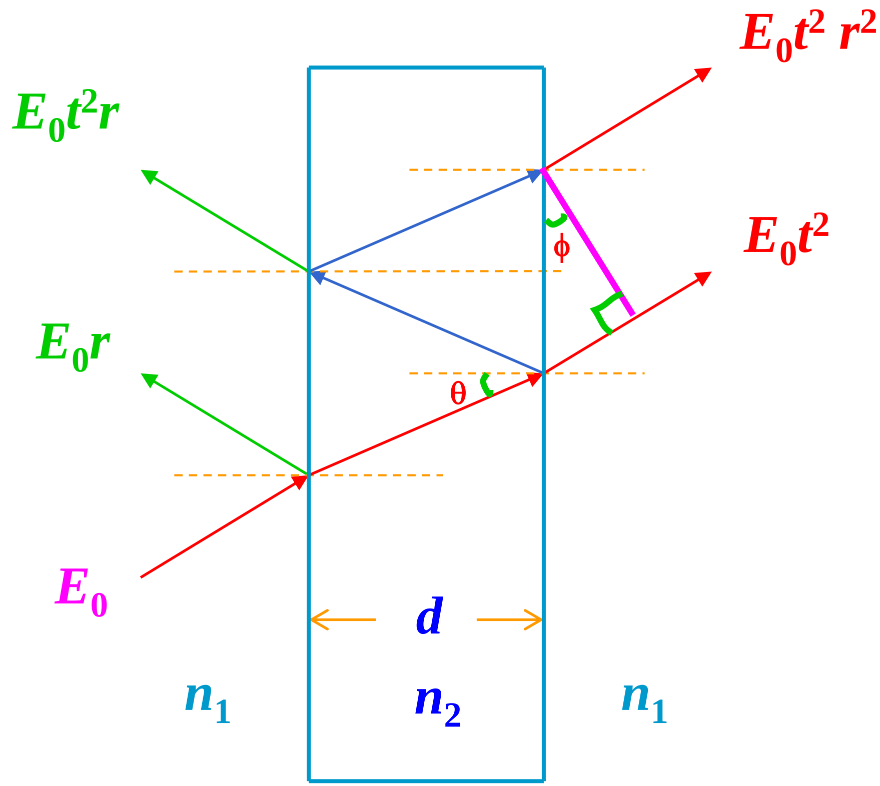
$$\Rightarrow E_T = \frac{E_0 t^2}{1 - r^2 e^{i\delta}}$$

$$\Rightarrow I_T \propto {E_T}^2 = {E_0}^2 \frac{t^4}{\left|1 - r^2 e^{i\delta}\right|^2}$$

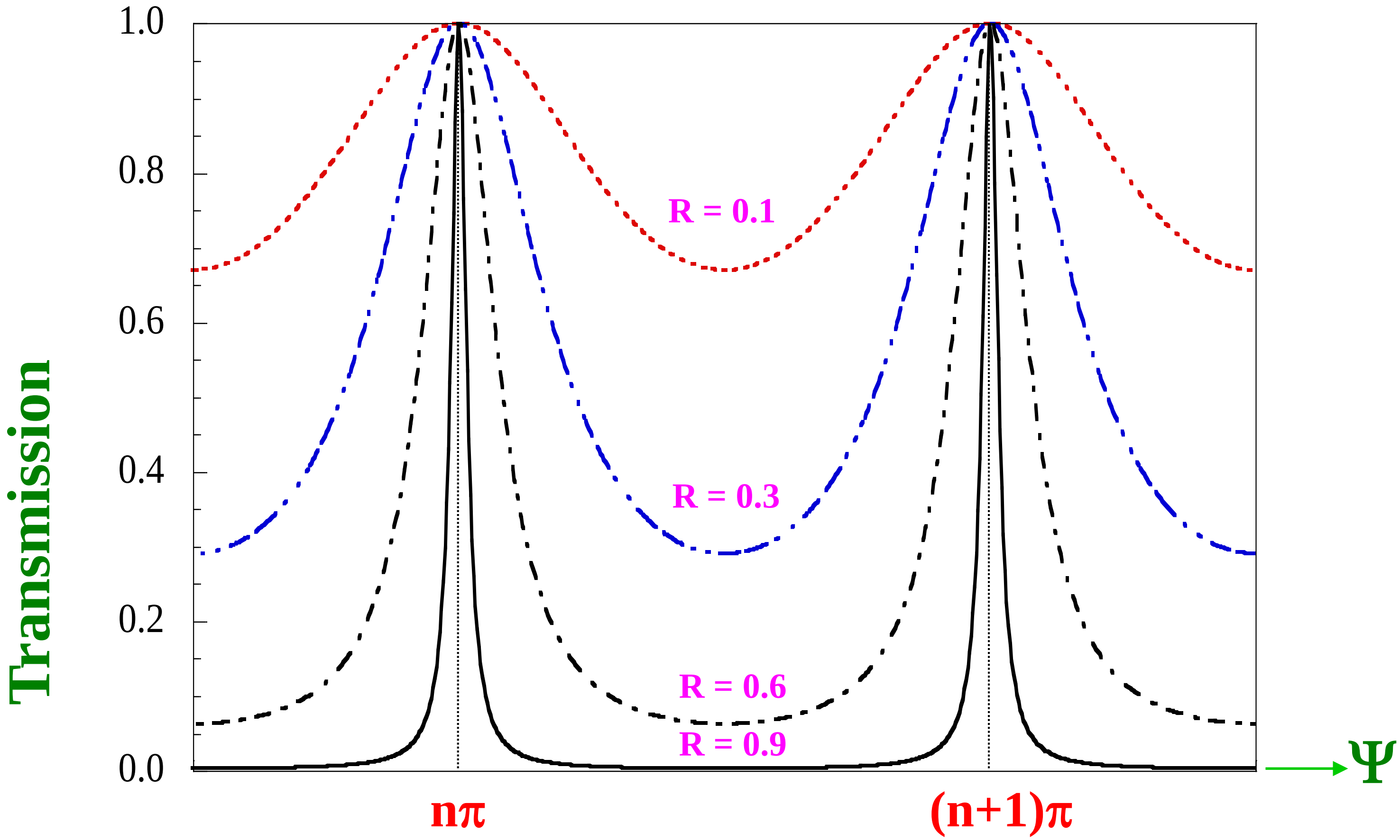
$$\Rightarrow T_{etalon} = \frac{I_T}{I_0} = \frac{{E_T}^2}{{E_0}^2} = \frac{T^2}{\left|1 - \mathrm{R} e^{i\delta}\right|^2}$$

$$\Rightarrow T_{etalon} = \frac{1}{1 + F\sin^2\Psi}$$

$$\text{where} \quad F = \frac{4R}{(1-R)^2}, \qquad \Psi = \frac{2\pi}{\lambda} d\cos\theta \quad (\lambda = \frac{\lambda_0}{n_2})$$



Transmission Function



$$T_{etalon} = \frac{1}{1 + F\sin^2\Psi} \quad where \quad F = \frac{4R}{(1-R)^2}, \quad \Psi = \frac{2\pi}{\lambda} d\cos\theta$$

Free Spectral Range of the Etalon

The wavelength (or frequency) difference between two passbands is called the "Free Spectral Range (FSR)" of the etalon.

$$\text{Since } T_{etalon} = \frac{1}{1+F\sin^2\Psi},$$

$$\text{maxima occur when } \Psi = \frac{2\pi}{\lambda}d = m\pi, \text{ or } 2nd\cos\theta = m\lambda_0$$

$$\text{Hence } m = \frac{2nd}{\lambda_0} \text{ and } m-1 = \frac{2nd}{(\lambda_0+\Delta\lambda)}$$

$$\Rightarrow 1 = 2nd\left(\frac{1}{\lambda_0} - \frac{1}{\lambda_0+\Delta\lambda}\right) \Rightarrow 1 \cong 2nd\frac{\Delta\lambda}{\lambda_0{}^2} \Rightarrow \Delta\lambda \cong \frac{\lambda_0{}^2}{2nd} \quad \cdots\cdots \text{ (1)}$$

$$\nu = \frac{c}{\lambda} \Rightarrow \Delta\nu\big|_{\lambda_0} = -\frac{c}{\lambda_0{}^2}\Delta\lambda \Rightarrow |\Delta\nu| = \frac{c}{2nd} \quad \cdots\cdots \text{ (2)}$$

Equations (1) and (2) give the variations which are required to move a passband by one order of magnitude.

Longitudinal Modes of a Laser Resonator

- **In a laser resonator the wavelength of the laser must satisfy the condition of a standing wave, i.e., nd = (λ_0/2)m, where m is an integer. This criterion is identical to that of the etalon if we set the angle θ equal to zero.**
- **Hence, if we consider only a single transverse mode, the separation of the longitudinal modes in a laser cavity is given by:**

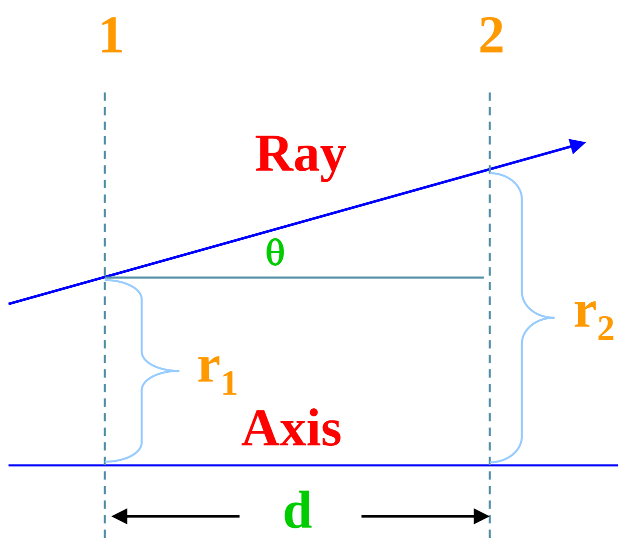
$$\Delta\lambda = \frac{\lambda_0^{\;2}}{2nd} \quad and \quad \Delta\nu = \frac{c}{2nd}$$

- **Example: *d* = 75 cm, λ_0 = 694.3 nm, then $\Delta\lambda$ = 0.0003 nm. For a ruby laser the linewidth is about 0.05 nm and, hence, there are about 160 longitudinal modes within this linewidth.**

ABCD Matrix for Free Space, Thin Lens, and Mirror

Refer to the figure, assuming all rays are paraxial, slope = r' = tanθ ≅ θ

$$r_2 = 1\cdot r_1 + d\cdot r_1' \qquad r_2' = 0\cdot r_1 + 1\cdot r_1' \quad \Rightarrow \quad \begin{bmatrix} r_2 \\ r_2' \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} r_1 \\ r_1' \end{bmatrix}$$



In general situation,

$$\begin{bmatrix} r_{out} \\ r_{out}' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} r_{in} \\ r_{in}' \end{bmatrix}$$

It can be shown that, for a

1) Length of Free Space

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

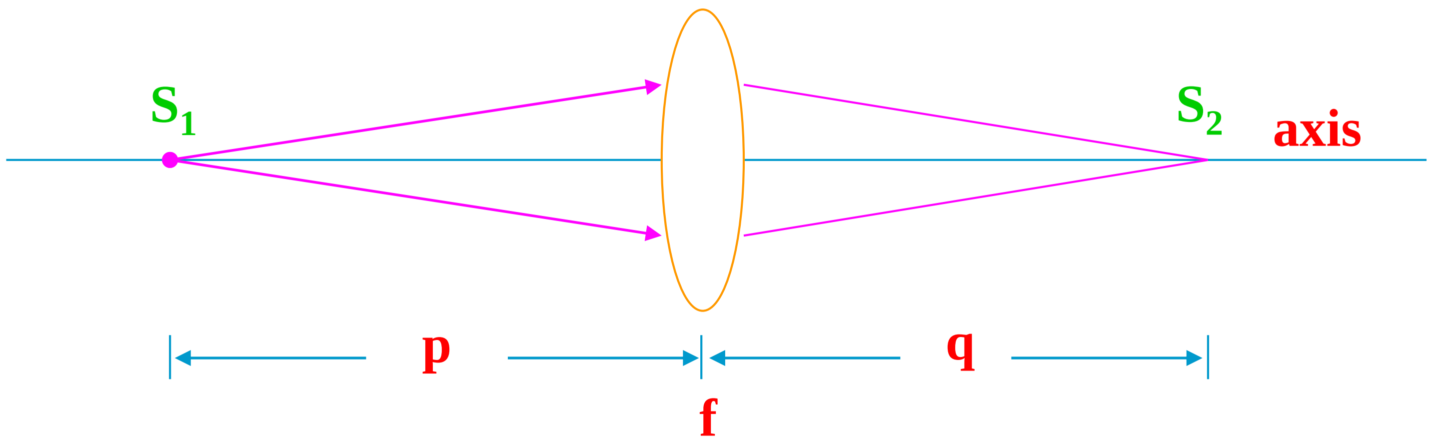
2) Thin Lens

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$

3) Mirror

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2/R & 1 \end{bmatrix}$$

Application of Ray Tracing



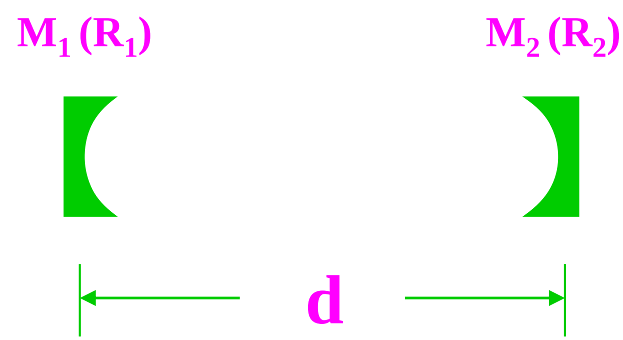
$$\begin{bmatrix} 0 \\ r_2' \end{bmatrix} = \begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ r_1' \end{bmatrix} = \begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & p \\ -1/f & 1-p/f \end{bmatrix} \begin{bmatrix} 0 \\ r_1' \end{bmatrix} = \begin{bmatrix} 1-q/f & p+q-pq/f \\ -1/f & 1-p/f \end{bmatrix} \begin{bmatrix} 0 \\ r_1' \end{bmatrix}$$

$$\Rightarrow 0 = (p+q-\frac{pq}{f})r_1' \Rightarrow p+q = \frac{pq}{f}$$

$$\Rightarrow \frac{1}{p} + \frac{1}{q} = \frac{1}{f}$$ **(Gaussian Lens Formula)**

Stability Criteria of a Laser Resonator

For a round trip, the transmission matrix:



$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & d \\ -1/f_1 & 1-d/f_1 \end{bmatrix} \begin{bmatrix} 1 & d \\ -1/f_2 & 1-d/f_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-d/f_2 & d+d(1-d/f_2) \\ -1/f_1-(1/f_2)(1-d/f_1) & (1-d/f_1)(1-d/f_2)-d/f_1 \end{bmatrix}$$

where $f_1 = R_1/2,\ f_2 = R_2/2$

It can be shown that the general condition for stability is as follows:

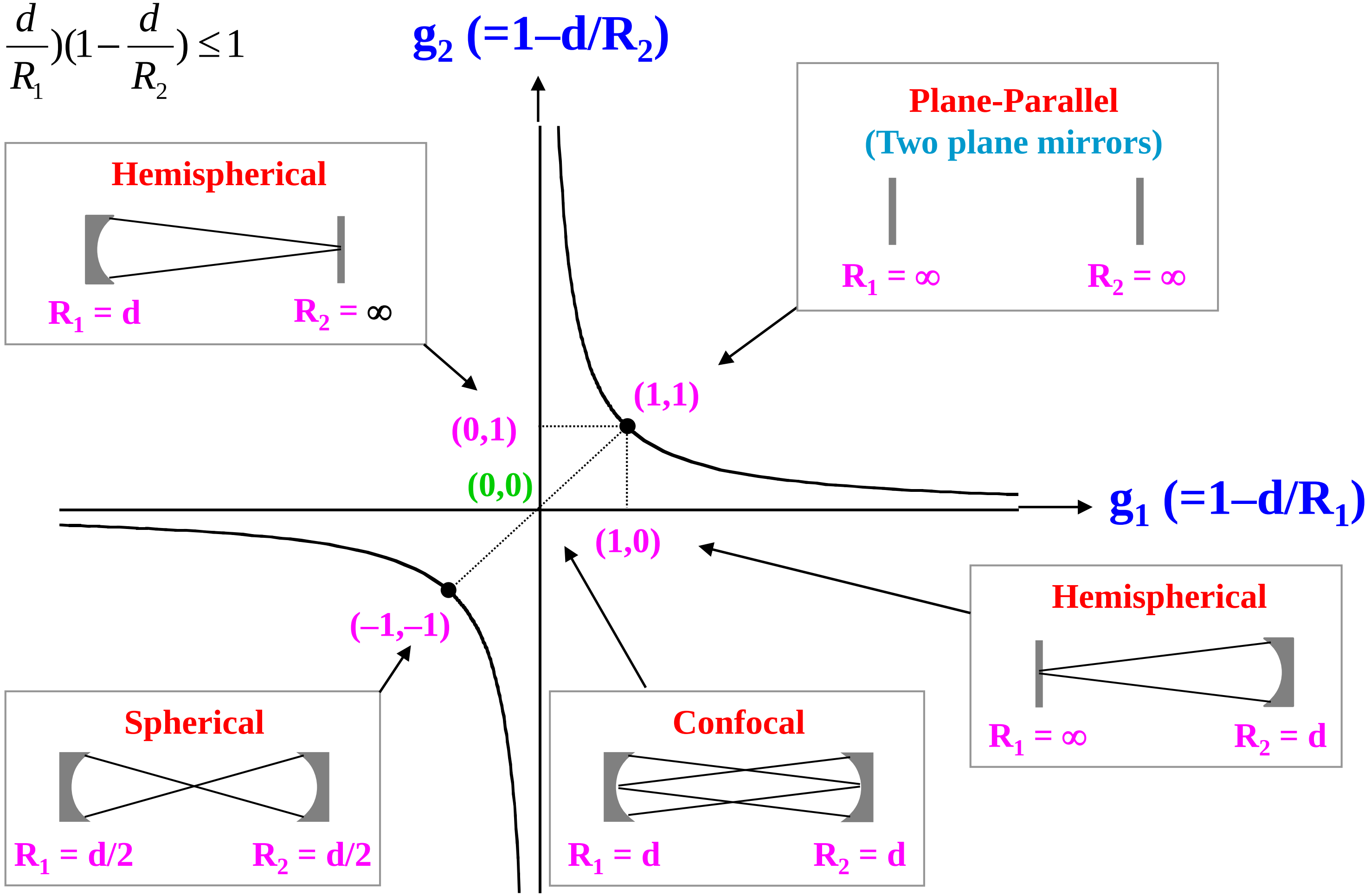
$$-1 \le \frac{A+D}{2} \le 1 \quad or \quad 0 \le \frac{A+D+2}{4} \le 1$$

$$\frac{A+D+2}{4} = \frac{1}{4}[1-\frac{d}{f_1}-\frac{d}{f_2}+(1-\frac{d}{f_1})(1-\frac{d}{f_2})+2] = 1-\frac{d}{2f_1}-\frac{d}{2f_2}+\frac{d^2}{4f_1f_2} = (1-\frac{d}{2f_1})(1-\frac{d}{2f_2}) = (1-\frac{d}{R_1})(1-\frac{d}{R_2})$$

Therefore, the stability condistion becomes: $0 \le (1-\frac{d}{R_1})(1-\frac{d}{R_2}) \le 1$, or equivalently, $0 \le g_1 g_2 \le 1$.

Stability Diagram for Laser Resonators

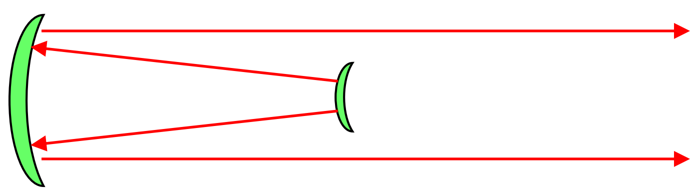
$$0 \le (1-\frac{d}{R_1})(1-\frac{d}{R_2}) \le 1$$



Remarks for Stability Diagram

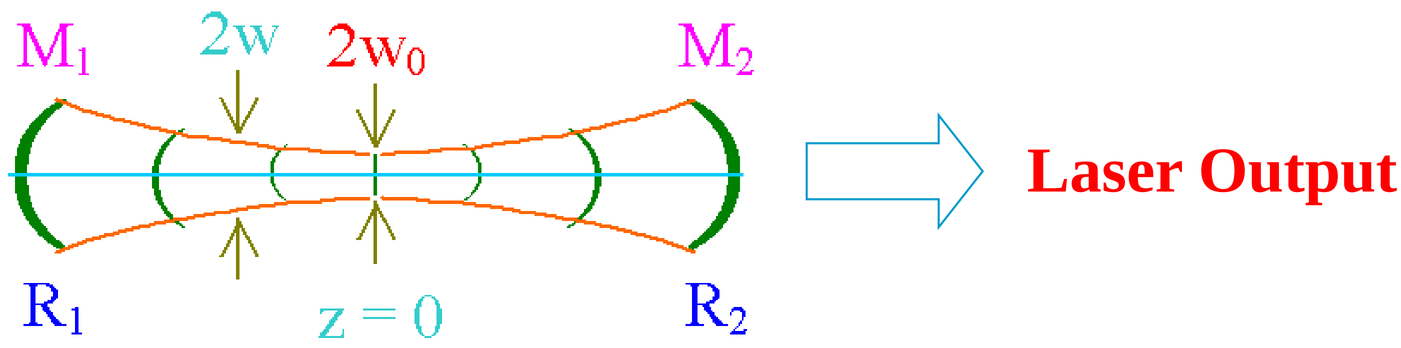
- In the stability diagram, laser resonator with its (g_1,g_2) parameters located within the area bounded by the two curves and the x-y axes is considered stable. It is conditionally stable if (g_1,g_2) is on the border. Under this circumstance, laser cavity requires perfect alignment in order to get laser action.
- Condition of unstable laser resonator:

$$\left(\frac{A+D}{2}\right)^2 > 1$$



- Unstable lasers are useful in high power applications (e.g., high power CO_2 lasers). For these specific systems the rays that walk off the mirrors constitute the output.

Gaussian Beams (TEM_{00} Mode)



For lowest-order TEM_{00} mode (in cylindrical coordinates):

$$E(x,y,z)=E_0\times\frac{w_0}{w(z)}\exp[-\frac{r^2}{w^2(z)}]\times\exp\{-j[kz-\tan^{-1}(\frac{z}{z_0})]\}\times\exp[-j\frac{kr^2}{2R(z)}]$$

where $$w^2(z)=w_0{}^2[1+(\frac{\lambda_0 z}{\pi n w_0{}^2})^2]=w_0{}^2[1+(\frac{z}{z_0})^2]$$

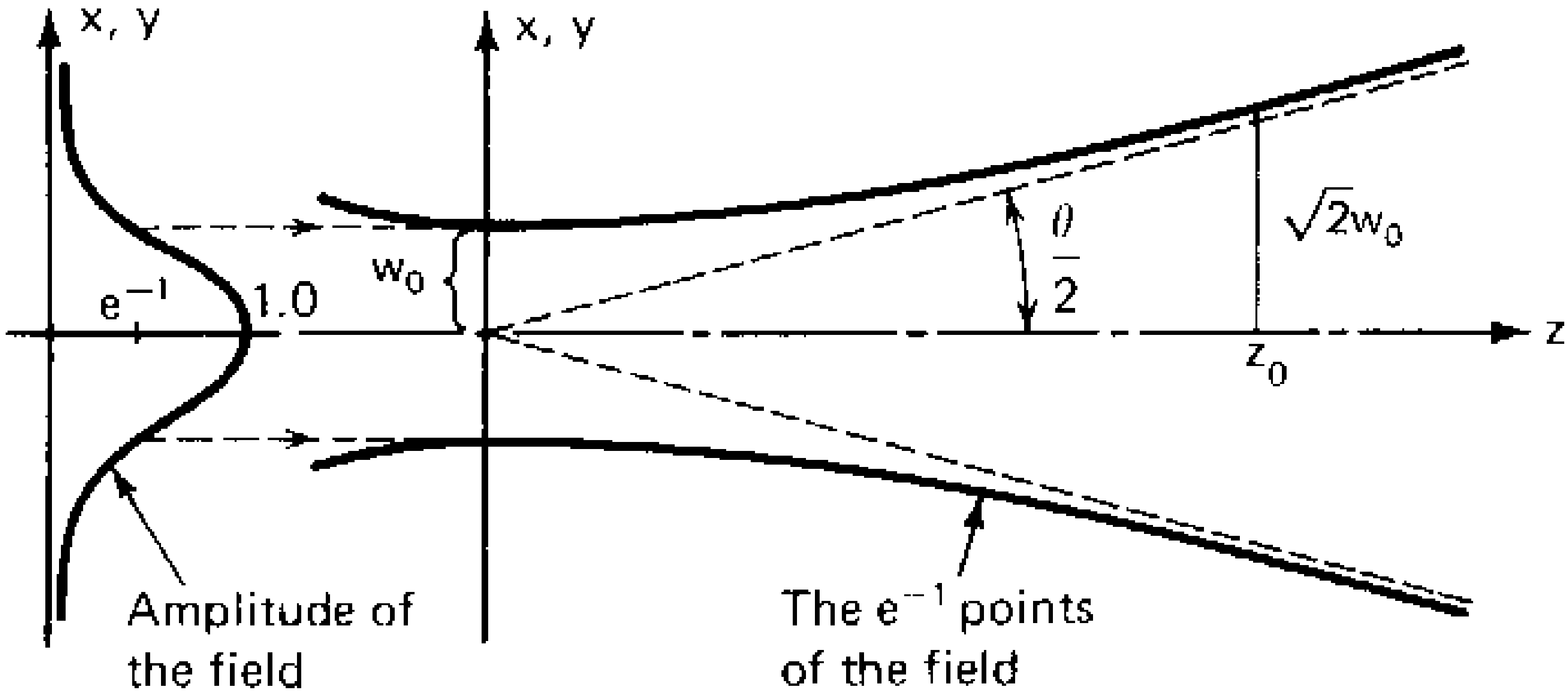
$\mathrm{w}=\sqrt{2}\mathrm{w}_0$ when $\mathrm{z}=\mathrm{z}_0$

$$R(z)=z[1+(\frac{\pi n w_0{}^2}{\lambda_0 z})]=z[1+(\frac{z_0}{z})^2]$$

R = ∞ at z = 0 (plane wave)
R = z when z → ∞
(i.e., like a spherical wave)

$$z_0=\frac{\pi n w_0{}^2}{\lambda_0}$$

Spreading of a TEM_{00} Mode

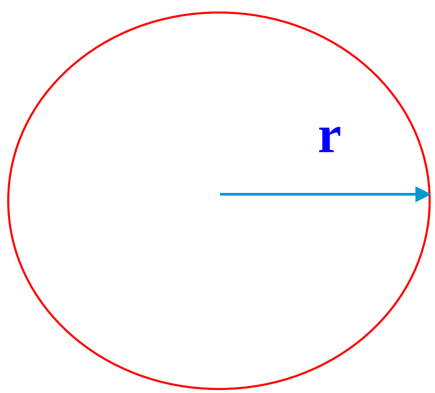


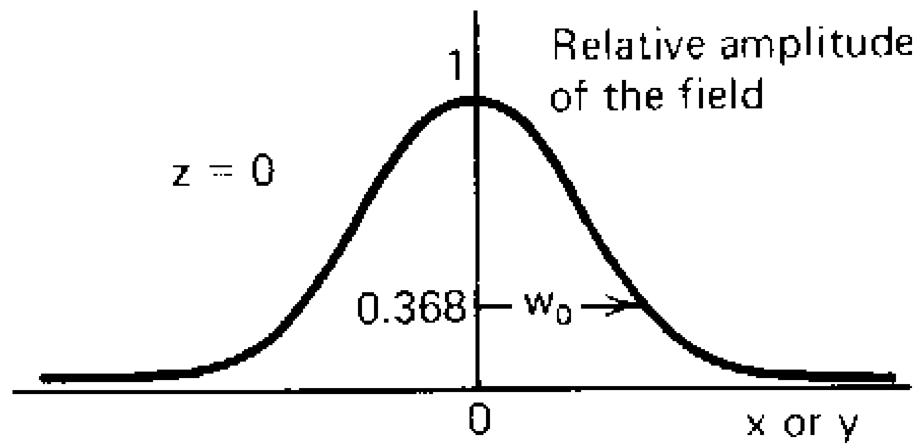
The expansion angle (full angle) of the laser beam is: $\theta \cong \dfrac{2\lambda_0}{\pi n w_0}$

A Few Notes for the Gaussian Beams

1) Amplitude of the field

$$\left|\frac{E(x,y,z)}{E_0}\right| = \frac{w_0}{w(z)}\exp[-(\frac{r}{w})^2] \Rightarrow \frac{E(r=w)}{E(r=0)} = e^{-1} = 0.368 = 36.8\%$$



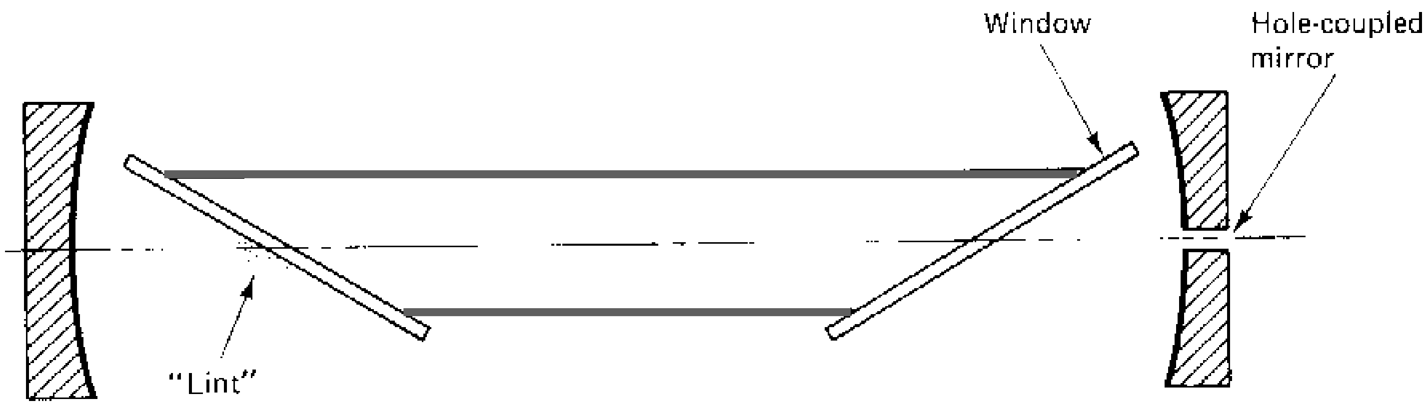


2) Radial phase factor, exp[–j(kr^2/2R(z))]

a) the plane z = constant is not an equiphase surface (i.e., the phase is a function of r)

b) the phase front is not planar; it is curved. (actually, R(z) is the radius of curvature)

Higher-Order Hermite-Gaussian Modes



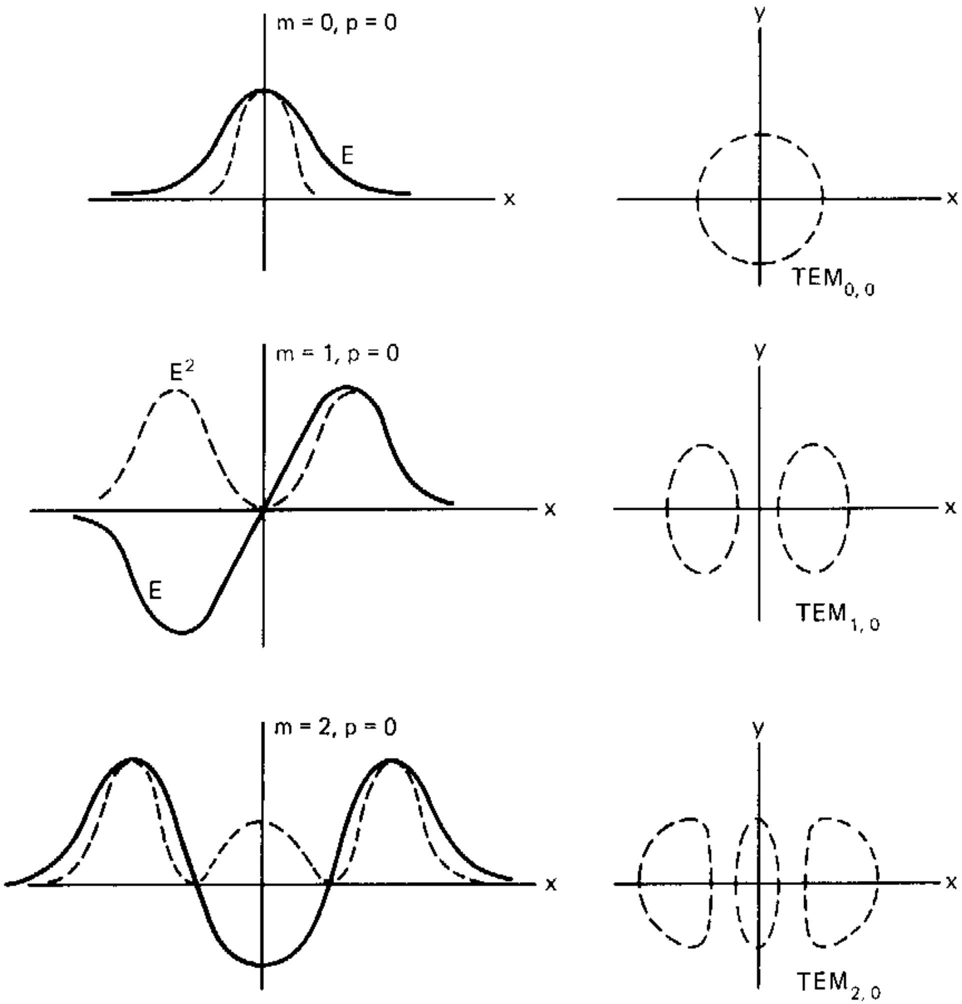
1) Higher-order modes:

$$E(x,y,z)=E_{m,p}\times H_m[\frac{\sqrt{2}x}{w(z)}]\times H_p[\frac{\sqrt{2}y}{w(z)}]\times\frac{w_0}{w(z)}\exp[-\frac{x^2+y^2}{w^2(z)}]$$

$$\times\exp\{-j[kz-(1+m+p)\tan^{-1}(\frac{z}{z_0})]\}\times\exp[-j\frac{kr^2}{2R(z)}]$$

2) All Hermite-Gaussian modes have same divergence angle $\theta = 2\lambda_0/\pi n w_0$

The Field E, Intensity E^2, and Dot Pattern of Various Modes



Hermite polynomials of order *m* and argument *u*:

$$H_m(u) = (-1)^m e^{u^2} \frac{d^m e^{-u^2}}{du^m}$$

Some of the lower-order Hermite polynomials:

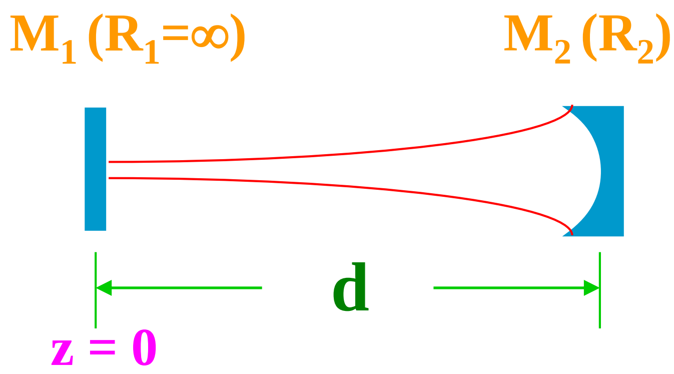
$H_0(u) = 1 \Rightarrow 1$

$H_1(u) = 2(u) \Rightarrow u$

$H_2(u) = 2(2u^2 - 1) \Rightarrow 2u^2 - 1$

Gaussian Beams in Simple Stable Resonators

Example: hemispherical resonator



$$R(z) = z[1+(\frac{z_0}{z})^2]$$

$$\Rightarrow R(d) = R_2 = d[1+(\frac{z_0}{d})^2]$$

$$\Rightarrow z_0 = \frac{\pi w_0^{\ 2}}{\lambda} = (dR_2)^{1/2}(1-\frac{d}{R_2})^{1/2} \cdots (1)$$

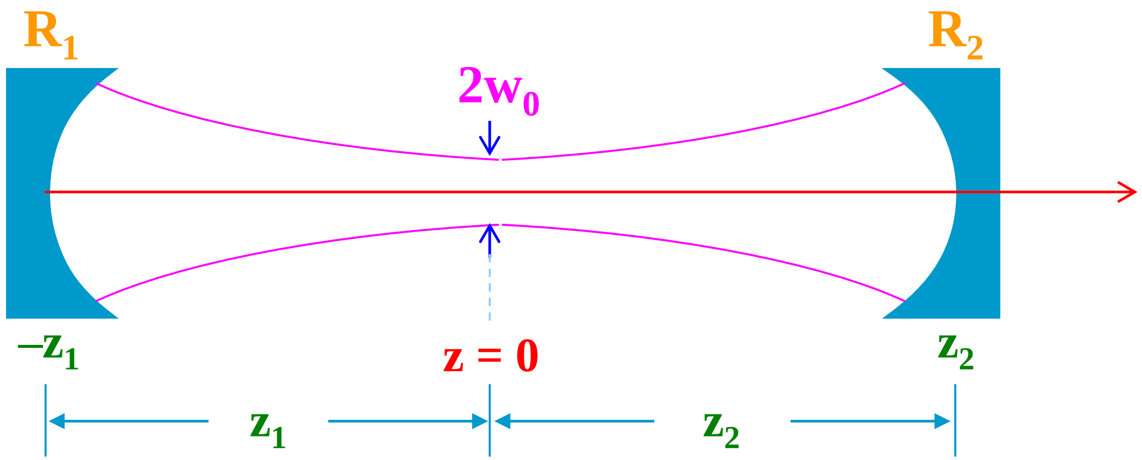
$$w^2(z) = w_0^{\ 2}[1+(\frac{z}{z_0})^2] \Rightarrow \frac{\pi w^2(d)}{\lambda} = \frac{\pi w_0^{\ 2}}{\lambda}[1+(\frac{d}{z_0})^2] \cdots (2)$$

Plug Eq. (1) into Eq. (2), we have

$$\frac{\pi w^2(d)}{\lambda} = (dR_2)^{1/2}(1-\frac{d}{R_2})^{1/2}[1+\frac{d^2}{dR_2(1-d/R_2)}]$$

$$\Rightarrow w(d) = (\frac{\lambda}{\pi})^{1/2}(\frac{dR_2}{1-d/R_2})^{1/4}$$

Regular Stable Laser Resonator



$$z_1 + z_2 = d \cdots (1)$$

$$R(z_2) = R_2 = z_2[1 + (\frac{z_0}{z_2})^2] \cdots (2) \qquad R(-z_1) = -R_1 = -z_1[1 + (\frac{z_0}{z_1})^2] \cdots (3)$$

Solve Eqs. (1), (2), and (3), we have

$$\text{a) } z_0^{\ 2} = (\frac{\pi w_0^{\ 2}}{\lambda_0})^2 = \frac{d(R_1 - d)(R_2 - d)(R_1 + R_2 - d)}{(R_1 + R_2 - 2d)^2}$$

$$\text{b) } z_1 = \frac{d(R_2 - d)}{R_1 + R_2 - 2d} \qquad \text{d) } w_1^{\ 4} = (\frac{\lambda R_1}{\pi})^2 \frac{R_2 - d}{R_1 - d} (\frac{d}{R_1 + R_2 - d})$$

$$\text{c) } z_2 = \frac{d(R_1 - d)}{R_1 + R_2 - 2d} \qquad \text{e) } w_2^{\ 4} = (\frac{\lambda R_2}{\pi})^2 \frac{R_1 - d}{R_2 - d} (\frac{d}{R_1 + R_2 - d})$$

Longitudinal and Transverse Laser Modes

- **Light emitted by most lasers contains several discrete optical frequencies, separated from each other by frequency differences that can be associated with different modes of the optical resonator.**
- **Longitudinal modes differ from one another in their frequency. The mode spacing $\Delta\nu$ is equal to c/2nd, where c is the speed of light, n and d are refractive index and cavity length, respectively. (note: $\Delta\lambda = \lambda^2/2nd$)**
- **Transverse modes (also called "lateral modes", or "spatial modes") differ from one another in their field distribution in a plane perpendicular to the direction of propagation.**
- **The fundamental transverse mode is TEM_{00} Gaussian mode. The spot size of the Gaussian beam (w) is defined as the radius at which the intensity of the TEM_{00} mode is $1/e^2$ of its peak value on the axis.**